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Metaplectic Representation of $osp_q(1/2)$ Algebra and Basic Hypergeometric Functions

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In this paper we use the bosonic realization of $osp_q(1/2)$ algebra to obtain its metaplectic representation. The group element for this algebra is shown to be described in terms of the basic hypergeometric function.

KEY WORDS: metaplectic representation; hypergeometric functions.

1. INTRODUCTION

Quantum groups or q-deformation of Lie algebra implies some specific deformation of classical Lie algebra and arise in many branches of physics and mathematics (Drinfeld, 1986; Faddeev *et al.*, 1988; Jimbo, 1985; Manin, 1988; Woronowics, 1987). From a mathematical point of view, it is a noncommutative associative quasi-triangular Hopf algebra. They offer the possibility of a group theoretical interpretation of the so-called basic hypergeometric series, quite in analogy with the Lie group interpretation of ordinary special functions (Miller, 1968; Vilenkin, 1968). Indeed, various calsses of q-orthonormal polynomials have already been identified in the representation theory of quantum groups (Masuda *et al.*, 1988, 1990; Vaskman and Soibelman, 1988). This group-theoretical setting enables us to find new properties of the q-special function. Recently, Floreanini and Vinet (1992a,b) discussed the connection of the metaplectic representation of the real form $su_q(1, 1)$ with a q-generalization of the Gegenbauer polynomials. In this paper we apply their method to the q-deformed-graded Lie algebra ($osp_q(1/2)$) and discuss the metaplectic representation of $osp_q(1/2)$ algebra. And we will show that the matrix elements of the certain operators of $osp_q(1/2)$ algebra are described in terms of the basic hypergeometric functions generalizing the Gegenbauer polynomial.

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2. THE METAPLECTIC REPRESENTATION OF OSP$_q$(1/2) ALGEBRA

We recall first that the osp$_q$(1/2) has three even (bosonic) generators H, $J_\pm$ generating a su$_q$(2) subalgebra, and two odd (fermionic) generators $v\pm$, with (anti)commutation relations (Saleur, 1990)

$$[H, v_\pm] = \pm\frac{1}{2}v\pm, \quad \{v_+, v_-\} = -\frac{1}{4}[2H], \tag{1}$$

where $[x]$ is defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

From the relation

$$J_\pm = \pm 4(v_\pm)^2, \tag{2}$$

the remaining (anti)commutation relations are easily obtained. An algebra homomorphism (coproduct) Δ : osp$_q$(1/2) $\rightarrow$ f osp$_q$(1/2) $\otimes$ osp$_q$(1/2) reads

$$\Delta(v_\pm) = q^H \otimes v_\pm + v_\pm \otimes q^{-H}, \quad \Delta(H) = id \otimes H + H \otimes id. \tag{3}$$

Similarly counit ϵ and antipod S are defined as

$$\epsilon(H) = \epsilon(v_\pm) = 0, \quad S(v_\pm) = -q^{\mp 1/2}v_\pm, \quad S(q^H) = q^{-H}. \tag{4}$$

One can easily check that the algebra (1) is endowed with a Hopf structure. In order to realize the osp$_q$(1/2) algebra, we introduce the following q-boson algebra:

$$aa^+ - qa^+a = 1, \quad [N, a^+] = a^+, \quad [N, a] = -a. \tag{5}$$

This algebra is realized in terms of the q-difference operator as follows:

$$N = z\partial_z, \quad a = \frac{1}{1-q}D, \quad a^+ = z, \tag{6}$$

where

$$D = \frac{1}{z}(1 - T).$$

The operator T is called scaling operator satisfying $Tf(z) = f(qz)$.

Then the bosonic realization of osp$_q$(1/2) algebra is given by

$$v_+ = \frac{1}{2\sqrt{[2]_{q^{1/2}}}}a^+$$

$$v_- = -\frac{1}{2\sqrt{[2]_{q^{1/2}}}}a$$

$$H = \frac{1}{2}(N + 1/2). \tag{7}$$

Using the difference operator realization of q-boson algebra we can obtain the difference operator realization of osp$_q$(1/2) algebra as follows:

$$
\begin{aligned}
v_+ &= \frac{1}{2\sqrt{[2]_{q^{1/2}}}} z \\
v_- &= -\frac{1}{2\sqrt{[2]_{q^{1/2}}}} \frac{1}{1-q} D \\
H &= \frac{1}{2}(z\partial_z + 1/2).
\end{aligned} \tag{8}
$$

In the limit $q \to 1$ this representation reduces to the metaplectic representation of osp(1/2) algebra.

3. GROUP ELEMENTS AND BASIC HYPERGEOMETRIC FUNCTION

The connection of the realization (8) with q-polynomials arises by considering the matrix elements of certain operators of the osp$_q$(1/2) algebra. In analogy with the ordinary osp(1/2) algebra let us introduce the elements

$$
\begin{aligned}
U(\alpha, \beta, \gamma) = {} & E_{q^{1/2}}(2\alpha(1-q^{1/2})\sqrt{[2]_{q^{1/2}}} v_+) \\
& \times E_{q^{1/2}}(-2\beta(1-q^{1/2})\sqrt{[2]_{q^{1/2}}} v_-) \\
& \times E_{q^{1/2}}(2\gamma(1-q^{1/2})H),
\end{aligned} \tag{9}
$$

where the q-exponential function is defined as

$$
E_{q^{1/2}}(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/4}}{(q^{1/2}; q^{1/2})_n} z^n \tag{10}
$$

and the q-shifted factorial is defined as

$$
\begin{aligned}
&(a;q)_n = (1-a)(1-qa)\cdots(1-aq^{n-1}), \quad n = 1, 2, \ldots \\
&(a;q)_0 = 1.
\end{aligned} \tag{11}
$$

Using the Eq. (8) we have

$$
\begin{aligned}
U(\alpha, \beta, \gamma) = {} & E_{q^{1/2}}(\alpha(1-q^{1/2})z) \\
& \times E_{q^{1/2}}(\beta(1+q^{1/2})^{-1} D) \\
& \times E_{q^{1/2}}(\gamma(1-q^{1/2})(z\partial_z + 1/2)).
\end{aligned} \tag{12}
$$

We define the matrix elements $U_{kn}(\alpha, \beta, \gamma)$ through

$$
U(\alpha, \beta, \gamma) z^n = \sum_{k=0}^{\infty} U_{kn}(\alpha, \beta, \gamma) z^k. \tag{13}
$$

Now we shall show that the matrix elements $U_{kn}(\alpha, \beta, \gamma)$ can be expressed in terms of the q-generalization of Gegenbauer polynomials. If we act the third part of Eq. (12) on the state z^n, we obtain

$$\begin{aligned} &E_{q^{1/2}}(\gamma(1-q^{1/2})(z\partial_z+1/2))z^n \\ &= E_{q^{1/2}}(\gamma(1-q^{1/2})(n+1/2))z^n. \end{aligned} \tag{14}$$

From the definition of the q-exponential one shows that

$$\begin{aligned} &E_{q^{1/2}}(\beta(1+q^{1/2})^{-1}D^+)z^n \\ &= \frac{\sum_{m=0}^{n} q^{m(m-1)/4}(1+q^{1/2})^{-m}(q;q)_n\beta^m}{(q^{1/2};q^{1/2})_{n-m}z^{n-m}.} \end{aligned} \tag{15}$$

One thus obtains

$$\begin{aligned} &E_{q^{1/2}}(\alpha(1-q^{1/2})z)E_{q^{1/2}}(\beta(1+q^{1/2})^{-1}D^+)z^n \\ &= \sum_{k=0}^{\infty} q^{\frac{1}{4}(n-k)(n-k-1)}(1+q^{1/2})^{k-n}\beta^{k-n}z^k \\ &\quad\times\left[\sum_{k=0}^{k}\frac{q^{\frac{1}{2}(l+n-k-1)}(q;q)_n}{(q^{1/2};q^{1/2})_l(q^{1/2};q^{1/2})_{n+l-k}(q;q)_{k-l}}\left(\frac{1-q^{1/2}}{1+q^{1/2}}\alpha\beta\right)^l\right]. \end{aligned} \tag{16}$$

With the help of the identity

$$(a;q)_{n-l} = \frac{(q;q)_n}{(q^{1-n}/a;q)_l}(-q/q)^l q^{l(l-1)/2-nl}$$

and

$$(q;q)_l = (q^{1/2};q^{1/2})_l(-q^{1/2};q^{1/2})_l$$

and reverting the direction of the inner summation, the term in the square bracket in Eq. (16) can be shown to be proportional to a basic hypergeometric function, defined by the series

$${}_2\phi_1(a,b;c,d;q^{1/2},x) = \sum_{n=0}^{\infty}\frac{(a;q^{1/2})_n(b;q^{1/2})_n}{(c;q^{1/2})_n(d;q^{1/2})_n}z^n. \tag{17}$$

Explicit computation yields

$$\begin{aligned} U_{kn}(\alpha,\beta,\gamma) = {}& E_{q^{1/2}}(\gamma(1-q^{1/2})(n+1/2))q^{\frac{1}{4}(n^2-n+k^2-k)} \\ &\times[(-q^{1/2};q^{1/2})_n/(q^{1/2};q^{1/2})_k]\alpha^k\beta^n(1-q^{1/2})^k(1+q^{1/2})^{-n} \\ &\times {}_2\phi_1\left(q^{-n/2},q^{-k/2};q^{1/2},-q^{1/2};q^{1/2},\frac{1+q^{1/2}}{1-q^{1/2}}(\alpha\beta)^{-1}\right). \end{aligned} \tag{18}$$

4. CONCLUSION

In this paper we use the bosonic realization of $\mathrm{osp_q}(1/2)$ algebra to obtain its metaplectic representation. Moreover the group element for this algebra is shown to be described in terms of the basic hypergeometric function. Of course we can obtain the fermionic realization of the $\mathrm{osp_q}(1/2)$ algebra, which is given by

$$v_+ = \frac{1}{2}a^+[2h - N - 1/2]_+ f + \frac{1}{2} f^+[2h - N], \tag{19}$$

$$v_- = \frac{1}{2}[N + 1/2]_+ f + \frac{1}{2} f^+ a \tag{20}$$

$$H = N + \frac{1}{2} f^+ f - h \tag{21}$$

where

$$aa^+ - qa^+a = q^{-N}, \quad [N, a^+] = a^+, [N, a] = -a \tag{22}$$

and

$$f, f^+ = 1, \quad f^2 = (f^+)^2 = 0. \tag{23}$$

However this realization looks too complicated, so we adopt the bosonic realization of the q-deformed-$\mathrm{osp_q}(1/2)$ algebrea. I think that it is very interesting to find the metaplectic representation for the fermionic realization and to derive the connection with some types of q-deformed polynomials. I hope that this work and its related topics will become clear in the near future.

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